U.G. 2nd Semester Examination - 2022

MATHEMATICS

[PROGRAMME]

Course Code : MATH-G-CC-T-02

Course Title : Calculus & Differential Equations

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The symbols and notations have their usual meanings.

1. Answer any **ten** questions: $2\times10=20$

a) Apply Euler's Theorem to show that $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=6f$ where $f(x, y)=x^4y^2\sin^{-1}\frac{y}{x}$.

b) Evaluate the following limit (if exist):

$$\lim_{x\to0}\frac{3x+|x|}{7x-5|x|}.$$

c) If $y=x^{n-1}\log x$ then prove that $y_n=\frac{(n-1)!}{x}$.

d) Prove that the curve $r^n=a^n\cos n\theta$ and $r^n=b^n\sin n\theta$ cut orthogonally.

e) A function $f:[0, 1]\to R$ is defined by

$$f(x) = x,\ x \text{ is rational in } [0,1]$$
$$= 1-x,\ x \text{ is irrational in } [0,1].$$

Show that f is continuous at $\frac{1}{2}$ and discontinuous at every other point in [0, 1].

f) If $y=\left(x+\sqrt{1+x^2}\right)^m$, find the value of $y_n(0)$.

g) If $f(x)=\sin x$ then prove that $\lim_{h\to0}\theta=\frac{1}{\sqrt{3}}$ where θ is given by $f(h)=f(0)+hf'(\theta h)$, $0<\theta<1$.

h) Let $a\in\mathbb{R}$ and a real function f be such that $f''(x)$ exists in [a – h, a + h] for some h> 0. Prove that $\frac{f(a+h)-2f(a)+f(a-h)}{h^2}=f''(c)$ for some $c\in$ [a – h, a + h].

i) If $S_n=\int_0^{\frac{\pi}{2}}\frac{\sin(2n-1)x}{\sin x}dx$, n being an integer, show that $S_{n+1}=S_n=\frac{\pi}{2}$.

j) If $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \, dx$, $n > 1$ being a positive integer, then show that

$$I_n + n(n-1) I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}.$$

k) Prove that $\left(\frac{1}{3x^3y^3}\right)$ is an integrating factor of

$$y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0.$$

l) Solve and find the singular solution of the differential equation

$$\sin\left(x\frac{dy}{dx}\right)\cos y = \cos\left(x\frac{dy}{dx}\right)\sin y + \frac{dy}{dx}.$$

m) Obtain the differential equation of all circles each of which touches the axis of x at the origin.

2. Answer any **four** questions: 5×4=20

a) If $\log y = \tan^{-1} x$ then show that

$$(1+x^2)\frac{d^{n+2}y}{dx^{n+2}} + (2nx + 2x - 1)\frac{d^{n+1}y}{dx^{n+1}} + n(n+1)\frac{d^n y}{dx^n} = 0.$$

b) If $f(x, y) = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$ when $xy \neq 0$, $-\frac{\pi}{2} \le \tan^{-1}\left(\frac{x}{y}\right) \le \frac{\pi}{2}$, and $f(x, 0) = f(0, y) = 0$, then show that $\frac{\partial^2 f}{\partial x \partial y}(0, 0) \neq \frac{\partial^2 f}{\partial y \partial x}(0, 0)$.

c) If u is a homogeneous function in x and y of degree n having continuous second order partial derivatives, prove that each of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ is a homogeneous function in x and y of degree (n – 1).

d) Obtain a reduction formula for $\int \sin^m x \cos^n x \, dx$, m, n being positive integers, greater than 1.

e) Solve $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = y^2 \log y$.

f) Let $f : [a, b] \to \mathbf{R}$ be a function such that $f^{(n-1)}$ is continuous in [a, b] and $f^{(n)}$ exists in (a, b). Show that there exists a $\theta \in (0, 1)$ such that

$$f(b) = f(a) + (b-a)f'(a) + \ldots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(b-1)^n}{n!}f^{(n)}[a + \theta(b-a)]$$

3. Answer any **two** questions: 10×2=20

a) i) Show that

$$\frac{v-u}{1+v^2} < tan^{-1} v - tan^{-1} u < \frac{v-u}{1+u^2},$$

if $0 < u < v$.

ii) Let $f: R \to R$ and $g: R \to R$ be two functions such that f(r) =g(r) = 0 where $r \in R$. Further consider $g'(r) \neq 0$. Then prove that $\lim_{x \to r} \frac{f(x)}{g(x)} = \frac{f'(r)}{g'(r)}$.

iii) If φ and ψ are both continuous in [a, b] and are both derivable in (a, b) and if φ' and ψ' never vanish, then prove that $\frac{\varphi(\xi)-\varphi'(a)}{\psi(b)-\psi'(\xi)} = \frac{\varphi'(a)}{\psi'(\xi)}$, $a < \xi < b$.

3+3+4

b) i) Determine a and b such that

$$\lim_{x \to 0} \frac{x(1+a\cos x) - b\sin x}{x^3} = 1.$$

ii) Evaluate: $\underset{x \to 0}{\mathrm{Lt}} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$.

iii) What is the altitude of a right circular cone of maximum volume that can be inscribed in a sphere of radius a?

3+3+4

c) i) Obtain a reduction formula for

$$\int \frac{dx}{(a+b\sin x)^n}.$$

ii) Solve by the method of variation of parameters:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \frac{e^{-x}}{x^2}.$$

iii) Evaluate $\frac{1}{D^2+3D+2} e^{e^x}$ where $D \equiv \frac{d}{dx}$.

4+4+2

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